

Linear independence measures for values of certain q -series

I. Rochev*

Abstract

We prove, in a quantitative form, linear independence results for values of a certain class of q -series, which generalize classical q -hypergeometric series. These results refine our recent estimates.

1 Main result

Let \mathbb{K} be an algebraic number field of degree $\varkappa = [\mathbb{K} : \mathbb{Q}]$, $\mathcal{M}_{\mathbb{K}}$ the set of all places of \mathbb{K} . For $v \in \mathcal{M}_{\mathbb{K}}$ we normalize the absolute value $|\cdot|_v$ as follows:

1. $|p|_v = p^{-1}$ for finite $v|p$;
2. $|x|_v = |x|$ for $x \in \mathbb{Q}$ if $v|\infty$.

Then for $\alpha \in \mathbb{K}^*$ we have the so-called product formula

$$\prod_{v \in \mathcal{M}_{\mathbb{K}}} |\alpha|_v^{\varkappa_v} = 1,$$

where $\varkappa_v = [\mathbb{K}_v : \mathbb{Q}_v]$ are the local degrees.

For a vector $\vec{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{K}^{n+1}$ its (projective) absolute (multiplicative) height $H(\vec{\alpha})$ is given by

$$H(\vec{\alpha}) = \prod_{v \in \mathcal{M}_{\mathbb{K}}} |\vec{\alpha}|_v^{\varkappa_v/\varkappa}, \quad |\vec{\alpha}|_v = \max\{|\alpha_0|_v, |\alpha_1|_v, \dots, |\alpha_n|_v\}.$$

(In fact, by the product formula, $H(\cdot)$ is well defined on the projective space \mathbb{KP}^n .) In particular, for $\alpha \in \mathbb{K}$ its absolute height is given by

$$H(\alpha) = H((1, \alpha)) = \prod_{v \in \mathcal{M}_{\mathbb{K}}} \max\{|\alpha|_v^{\varkappa_v/\varkappa}, 1\}.$$

*Research is supported by RFBR (grant No. 09-01-00371a).

In view of the product formula, for $\alpha \in \mathbb{K}^*$ and any $v \in \mathcal{M}_{\mathbb{K}}$ we have the so-called fundamental inequality

$$|\log |\alpha|_v| \leq \frac{\varkappa}{\varkappa_v} \log H(\alpha).$$

Suppose $q \in \mathbb{K}$ and $w \in \mathcal{M}_{\mathbb{K}}$ satisfy $|q|_w > 1$ and $|q|_v \leq 1$ for all $v \in \mathcal{M}_{\mathbb{K}} \setminus \{w\}$. Further assume that polynomials $P(x, y) \in \mathbb{K}[x, y]$, $Q(x) \in \mathbb{K}[x]$ satisfy $d := \deg_y P(x, y) \geq 1$ and $P(n, q^n)Q(n) \neq 0$ for all $n \in \mathbb{Z}_{>0}$. Put

$$\Pi_n(z) = \prod_{k=1}^n P(k, z^k)/Q(k) \quad (n \in \mathbb{Z}_{\geq 0}) \quad (1.1)$$

and consider the function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Pi_n(q)}, \quad z \in \mathbb{C}_w,$$

where \mathbb{C}_w is the completion of the algebraic closure of \mathbb{K}_w .

The function $f(z)$ is entire. Indeed, let

$$P(x, y) = \sum_{\nu=0}^d p_{\nu}(x)y^{\nu};$$

then for all sufficiently large $n \in \mathbb{Z}_{>0}$ we have

$$|p_d(n)|_w \geq H(p_d(n))^{-\varkappa/\varkappa_w} \geq n^{-c}, \quad c = \text{const.}$$

Hence for large n we have

$$|P(n, q^n)|_w \geq \frac{1}{2} n^{-c} |q|_w^{dn}$$

and the assertion follows.

In this note we prove the following theorem.

Theorem 1. *Assume that the polynomials $P(x, y), Q(x)$ satisfy (at least) one of the following two conditions:*

- (a) $p_d(x)$ does not depend on x , or
- (b) $p_0(x)$ and $Q(x)$ do not depend on x .

Let $m \in \mathbb{Z}_{>0}$, $d_0 \in \mathbb{Z}_{\geq d}$. Suppose numbers $\alpha_j \in \mathbb{K}^*$ and $s_{j,k} \in \mathbb{Z}_{>0}$ ($1 \leq j \leq m$, $0 \leq k < d_0$) satisfy the following three conditions:

- (i) $\alpha_i \alpha_j^{-1} \notin q^{\mathbb{Z}}$ for $i \neq j$,
- (ii) $s_{j,k} \leq \deg p_d(x)$ for $1 \leq j \leq m$ and $d \leq k < d_0$, and

(iii) if $\deg p_0(x) = \deg Q(x)$, then $\alpha_j \notin (a/b)q^{\mathbb{Z}_{>0}}$ for all j , where a and b are the leading coefficients of $p_0(x)$ and $Q(x)$, respectively.

Then the numbers

$$1, f^{(\sigma)}(\alpha_j q^k) \quad (1 \leq j \leq m, 0 \leq k < d_0, 0 \leq \sigma < s_{j,k})$$

are linearly independent over \mathbb{K} . Moreover, there exist (effective) positive constants $C_0 = C_0(P, Q, q)$ and $H_0 = H_0(P, Q, q, m, d_0, \alpha_j, s_{j,k})$ such that for any vector $\vec{A} = (A_0, A_{j,k,\sigma}) \in \mathbb{K}^{1+\sum_{j,k} s_{j,k}} \setminus \{\vec{0}\}$ we have

$$\left| A_0 + \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} A_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) \right|_w \geq |\vec{A}|_w \exp(-C_0 m^{2/3} (\log H)^{4/3}),$$

where $H = \max\{H(\vec{A}), H_0\}$.

In the case $\mathbb{K} = \mathbb{Q}$, $|\cdot|_w = |\cdot|$, $Q(x) = 1$ the qualitative part of Theorem 1 was essentially proved by Bézivin [1]. Moreover, Bézivin's result implies that in this case the corresponding assertion is valid (with $d_0 = 1$ and slightly more restrictive conditions posed on α_j) even if the polynomial $P(x, y)$ does not satisfy conditions (a)–(b) of Theorem 1.

Recently the author [2] proposed a quantitative variant of Bézivin's method; in particular, a weak version of Theorem 1 was proved, with the estimate of the form $\exp(-C_0 m (\log H)^2)$. A modification of this method was proposed in [3] for the case when the polynomials $P(x, y), Q(x)$ do not depend on x . In this case a much stronger result than Theorem 1 is valid: the estimate for the linear form is polynomial in H and the conditions posed on q can be weakened. In [3] for simplicity only the case $\mathbb{K} = \mathbb{Q}$, $|\cdot|_w = |\cdot|$ was considered but extension to the general case is straightforward (cf., e. g., [4]).

Note that Theorem 1 allows one to describe all linear dependences (over \mathbb{K}) among values of the function $f(z)$ and its derivatives at points of the field \mathbb{K} (if the number q and the polynomials $P(x, y), Q(x)$ satisfy the aforementioned conditions). Indeed, the function $f(z)$ satisfies the functional equation

$$P\left(z \frac{d}{dz}, D_q\right)(f(z)) = P(0, 1) + Q\left(z \frac{d}{dz}\right)(zf(z)), \quad D_q f(z) := f(qz), \quad (1.2)$$

therefore, for any $\alpha \in \mathbb{K}^*$ and $s \geq \deg p_d(x)$ the number $f^{(s)}(\alpha)$ can be expressed as a linear combination of the numbers $1, f^{(\sigma)}(\alpha)$ with $0 \leq \sigma < \deg p_d(x)$, and $f^{(\sigma)}(\alpha q^{-\nu})$ with $1 \leq \nu \leq d$ and $\sigma \geq 0$. It follows that, given numbers $\beta_1, \dots, \beta_l \in \mathbb{K}^*$ and $t \in \mathbb{Z}_{>0}$, there exist α_j and $s_{j,k}$ satisfying the conditions of Theorem 1 such that the numbers $f^{(\tau)}(\beta_j)$ ($1 \leq j \leq l, 0 \leq \tau < t$) can be expressed as linear combinations of $1, f^{(\sigma)}(\alpha_j q^k)$ ($1 \leq j \leq m, 0 \leq k < d_0, 0 \leq \sigma < s_{j,k}$). Hence

any relation

$$B_0 + \sum_{j=1}^l \sum_{\tau=0}^{t-1} B_{j,\tau} f^{(\tau)}(\beta_j) = 0$$

can be rewritten in the form

$$A_0 + \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} A_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0,$$

where $A_0, A_{j,k,\sigma}$ are certain linear combinations of $B_0, B_{j,\tau}$. It follows from Theorem 1 that the coefficients $B_0, B_{j,\tau}$ must satisfy the system of linear equations $A_0 = A_{j,k,\sigma} = 0$. In other words, all linear dependences among values of the function $f(z)$ and its derivatives at points of the field \mathbb{K} follow from the functional equation (1.2).

Theorem 1 is a simple consequence of results of [2]. Since the case $\deg_x P(x, y) = \deg Q(x) = 0$ was considered in [3], in what follows we assume that $\deg_x P(x, y) + \deg Q(x) > 0$. In Section 2 we summarize the required results from [2]. In Section 3 we use them to construct auxiliary linear forms. In the final section Theorem 1 is proved.

2 Summary

Let $m \in \mathbb{Z}_{>0}$, $d_0 \in \mathbb{Z}_{\geq d}$, $\alpha_j \in \mathbb{K}^*$ ($1 \leq j \leq m$), $s_{j,k} \in \mathbb{Z}_{>0}$ ($1 \leq j \leq m$, $0 \leq k < d_0$). By \vec{x} denote the vector of variables $\vec{x} = (x_0, x_{j,k,\sigma})$, where $1 \leq j \leq m$, $0 \leq k < d_0$, $0 \leq \sigma < s_{j,k}$. Furthermore, put

$$s_j = \max_{0 \leq k < d_0} s_{j,k} \quad (1 \leq j \leq m),$$

$$S = s_1 + \dots + s_m.$$

Consider the polynomials

$$u_n = u_n(z, \vec{x}) = \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} \sigma! \binom{n}{\sigma} (\alpha_j z^k)^{n-\sigma} x_{j,k,\sigma} \in \mathbb{K}[z, \vec{x}], \quad (2.1)$$

$$v_n = v_n(z, \vec{x}) = \Pi_n(z) \cdot \left(x_0 + \sum_{k=0}^n \frac{u_k(z, \vec{x})}{\Pi_k(z)} \right) \in \mathbb{K}[z, \vec{x}], \quad (2.2)$$

where $\Pi_n(z)$ is given by (1.1).

2.1 First case

Suppose the polynomial $P(x, y)$ satisfies condition (a) of Theorem 1, i. e.,

$$P(x, y) = p_d y^d + \sum_{\nu=0}^{d-1} p_\nu(x) y^\nu, \quad p_d \in \mathbb{K}^*.$$

Put $h = \deg Q(x)$,

$$g_1 = \max \left\{ \max_{1 \leq \nu \leq d} \frac{\deg p_{d-\nu}(x)}{\nu}, \frac{h}{d} \right\} > 0.$$

Further, let \mathcal{B} be the backward shift operator given by

$$\mathcal{B}(\xi(n)) = \xi(n-1).$$

For $1 \leq j \leq m$ and $k \in \mathbb{Z}$ introduce the difference operator

$$\mathcal{A}_{k,j} = \mathcal{I} - \alpha_j q^k \mathcal{B},$$

where \mathcal{I} is the identity operator, $\mathcal{I}(\xi(n)) = \xi(n)$.

Finally, for $l \geq 0$, $n \geq (S + mh)l + m \sum_{k=0}^{l-1} \lfloor g_1 k \rfloor$ put

$$v_{l,n}(\vec{x}) = \prod_{k=0}^{l-1} \prod_{j=1}^m \mathcal{A}_{d_0-1-d-k,j}^{s_j+h+\lfloor g_1 k \rfloor} (v_n(q, \vec{x})) := \left(\prod_{k=0}^{l-1} \prod_{j=1}^m \mathcal{A}_{d_0-1-d-k,j}^{s_j+h+\lfloor g_1 k \rfloor} \right) (v_n(q, \vec{x})) \in \mathbb{K}[\vec{x}].$$

Then we have the following lemma.

Lemma 1. Assume that $\vec{\omega} = (\omega_0, \vec{\omega}_1) = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}_w^{1+\sum_{j,k} s_{j,k}}$ satisfies

$$\omega_0 + \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0.$$

Then for all $l \geq 0$, $n \geq (S + mh)l + m \sum_{k=0}^{l-1} \lfloor g_1 k \rfloor$ we have the estimate

$$|v_{l,n}(\vec{\omega})|_w \leq |\vec{\omega}_1|_w |q|_w^{-ln+mg_1 l^3/6+c(n+1)},$$

where the constant $c > 0$ depends only on the polynomials P, Q and the numbers $q, m, d_0, \alpha_j, s_{j,k}$.

Proof. See [2, Lemma 3.2]. □

Remark 1. Note that an estimate of the form

$$|v_{l,n}(\vec{\omega})|_w \ll_l |\vec{\omega}_1|_w |q|_w^{-ln+cn}$$

is trivial. Indeed, we have

$$v_n(q, \vec{\omega}) = - \sum_{j=n+1}^{\infty} \frac{u_j(q, \vec{\omega})}{\prod_{k=n+1}^j P(k, q^k)/Q(k)} = \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} \omega_{j,k,\sigma} A_{j,k,\sigma},$$

where every $A_{j,k,\sigma}$ has an asymptotic expansion of the form

$$A_{j,k,\sigma} \sim \sum_{i=0}^{\infty} P_i(n) (\alpha_j q^{k-d-i})^n \quad \text{as } n \rightarrow \infty,$$

with $P_i(z) \in \mathbb{K}[z]$, $\deg P_i \leq \sigma + h + g_1 i$. The required estimate follows immediately.

2.2 Second case

In this subsection we assume that the polynomials $p_0(x) = p_0$ and $Q(x) = 1$ do not depend on x . Put

$$\varepsilon_0 = \begin{cases} 0 & \text{if } p_0 = 0, \\ 1 & \text{if } p_0 \neq 0, \end{cases}$$

$$g_2 = \max_{1 \leq \nu \leq d} \frac{\deg p_\nu(x)}{\nu} > 0.$$

For $0 \leq j \leq m$ and $k \in \mathbb{Z}$ introduce the difference operator

$$\mathcal{A}_{k,j} = \begin{cases} \mathcal{I} - p_0 z^k \mathcal{B} & \text{if } j = 0, \\ \mathcal{I} - \alpha_j z^k \mathcal{B} & \text{if } 1 \leq j \leq m, \end{cases}$$

where \mathcal{I}, \mathcal{B} are same as above. Note that if $p_0 = 0$, then $\mathcal{A}_{k,0} = \mathcal{I}$.

For $l \geq 0$, $n \geq (S + \varepsilon_0)l + (m + \varepsilon_0) \sum_{k=0}^{l-1} \lfloor g_2 k \rfloor$ put

$$v_{l,n} = v_{l,n}(z, \vec{x}) = \prod_{k=0}^{l-1} \prod_{j=0}^m \mathcal{A}_{k,j}^{s_j + \lfloor g_2 k \rfloor} (v_n) \in \mathbb{K}[\vec{x}][z], \quad s_0 := 1.$$

Define the (z) -order of a formal Laurent series $\xi(z) = \sum_{n \in \mathbb{Z}} a_n z^n \neq 0$ as

$$\text{ord}_z \xi(z) = \min\{n \mid a_n \neq 0\};$$

furthermore, put $\text{ord}_z 0 = +\infty$.

Then the following assertion holds.

Lemma 2. *For all $l \geq 0$, $n \geq (S + \varepsilon_0)l + (m + \varepsilon_0) \sum_{k=0}^{l-1} \lfloor g_2 k \rfloor$ we have*

$$\text{ord}_z v_{l,n} \geq ln - (m + \varepsilon_0)g_2 l^3 / 6 - c(n + 1),$$

where the constant $c > 0$ depends only on the polynomial P and the numbers $m, d_0, s_{j,k}$.

Proof. See [2, Lemma 3.3]. □

2.3 Non-vanishing lemma

For $n \geq 1$ put

$$V_n = V_n(z, \vec{x}) = \det(v_{i+j})_{i,j=0}^{n-1} \in \mathbb{K}[z, \vec{x}].$$

Then we have the following non-vanishing lemma.

Lemma 3. Assume that the polynomials P, Q and the numbers $\alpha_j, s_{j,k}$ satisfy the conditions of Theorem 1, $\vec{\omega} \in \mathbb{C}_w^{1+\sum_{j,k} s_{j,k}} \setminus \{\vec{0}\}$. Then for any $n_0 \in \mathbb{Z}_{>0}$ there is an integer n within the range $n_0 \leq n \leq c_1 n_0 + c_0$ such that $V_n(q, \vec{\omega}) \neq 0$, where $c_1 = c_1(P, Q), c_0 = c_0(Q, m, d_0, s_{j,k})$ are certain positive constants.

Proof. See [2, Lemma 4.3]. □

3 Main proposition

We begin with some notation. Suppose

$$A(\vec{z}) = A(z_1, \dots, z_n) = \sum_{\vec{v}} A_{\vec{v}} z_1^{\nu_1} \dots z_n^{\nu_n} \in \mathbb{K}[\vec{z}]$$

is a polynomial; then for $v \in \mathcal{M}_{\mathbb{K}}$ we put

$$|A|_v = \begin{cases} \sum_{\vec{v}} |A_{\vec{v}}|_v & \text{if } v | \infty, \\ \max_{\vec{v}} |A_{\vec{v}}|_v & \text{if } v \nmid \infty. \end{cases}$$

Furthermore, put

$$H(A) = \prod_{v \in \mathcal{M}_{\mathbb{K}}} |A|_v^{\varkappa_v / \varkappa},$$

$$H_w(A) = \prod_{v \in \mathcal{M}_{\mathbb{K}} \setminus \{w\}} |A|_v^{\varkappa_v / \varkappa}.$$

In the following proposition we assume the hypotheses of Theorem 1; we also keep the notation from the previous section.

Proposition 1. *There exists a constant $g_0 = g_0(P, Q) \in \mathbb{Z}_{>0}$ such that for any positive integers l, n with $n \geq mg_0 l^2 + Sl$ there is a linear form $L_{l,n}(\vec{x}) \in \mathbb{K}[\vec{x}]$ satisfying the following three conditions:*

1. *For any $\vec{\omega} = (\omega_0, \vec{\omega}_1) = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}_w^{1+\sum_{j,k} s_{j,k}}$ such that*

$$\omega_0 + \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0$$

we have

$$|L_{l,n}(\vec{\omega})|_w \leq |\vec{\omega}_1|_w |q|_w^{-ln+mg_0 l^3/2+O(n)}.$$

2. *The following estimates are valid:*

$$H(L_{l,n}) \leq H(q)^{dn^2/2+O(n^{3/2})},$$

$$H_w(L_{l,n}) \leq H(q)^{O(n \log n)}.$$

3. For any $\vec{\omega} \in \mathbb{C}_w^{1+\sum_{j,k} s_{j,k}} \setminus \{\vec{0}\}$ and $l_0, n_0 \in \mathbb{Z}_{>0}$ with $n_0 \geq mg_0 l_0^2 + Sl_0$ there exists an integer n with $n_0 \leq n \leq g_0 n_0 + O(1)$ such that $L_{l_0, n}(\vec{\omega}) \neq 0$.

The constants in the Landau symbols $O(\cdot)$ depend only on $P, Q, q, m, d_0, \alpha_j, s_{j,k}$.

Remark 2. In fact, the inequality $n \leq g_0 n_0 + O(1)$ in condition 3 of Proposition 1 can be replaced by $n \leq n_0 + O(l_0)$ (cf., e. g., [3, Section 3]).

The proof of the proposition is divided into two parts according to whether the polynomials P, Q satisfy condition (a) or (b) of Theorem 1. Before we proceed let us make some preliminary remarks.

Without loss of generality we can assume that $Q(x) \in \mathbb{Z}_{\mathbb{K}}[x]$. Furthermore, let $D \in \mathbb{Z}_{>0}$ be a general denominator of the numbers α_j and coefficients of the polynomial $P(x, y)$. Put

$$I_n = D^n \prod_{k=1}^n Q(k) \in \mathbb{Z}_{\mathbb{K}} \setminus \{0\}.$$

It follows from (2.1)–(2.2) that $I_n v_n \in \mathbb{Z}_{\mathbb{K}}[z, \vec{x}]$. Moreover, v_n is homogeneous in \vec{x} with $\deg_{\vec{x}} v_n = 1$, $\deg_z v_n = dn^2/2 + O(n)$.

3.1 Case (a)

We keep the notation from Subsection 2.1. Let us show that we can take

$$L_{l,n}(\vec{x}) = q^{\sum_{k=0}^{l-1} \sum_{j=1}^m (s_j + h + \lfloor g_1 k \rfloor)(k+1)} v_{l,n}(\vec{x}).$$

It follows from Lemma 1 that condition 1 of the proposition holds (provided that g_0 is large enough).

Further, since

$$L_{l,n}(\vec{x}) = \prod_{k=0}^{l-1} \prod_{j=1}^m (q^{k+1} \mathcal{I} - \alpha_j q^{d_0-d} \mathcal{B})^{s_j + h + \lfloor g_1 k \rfloor} (v_n(q, \vec{x})),$$

it is readily seen that $I_n L_{l,n}(\vec{x}) = A(q, \vec{x})$ for some polynomial $A(z, \vec{x}) \in \mathbb{Z}_{\mathbb{K}}[z, \vec{x}]$ with $\deg_z A \leq dn^2/2 + mg_1 l^3/3 + O(n) = dn^2/2 + O(n^{3/2})$. Hence for finite $v \in \mathcal{M}_{\mathbb{K}}$ we have

$$|L_{l,n}|_v \leq |I_n|_v^{-1} \max\{|q|_v, 1\}^{dn^2/2 + O(n^{3/2})}.$$

For a polynomial $A(\vec{z}) = A(z_1, \dots, z_n) = \sum_{\vec{\nu}} A_{\vec{\nu}} z_1^{\nu_1} \dots z_n^{\nu_n} \in \mathbb{K}[\vec{z}]$ put

$$\mathcal{L}(A) = \sum_{\vec{\nu}} \overline{|A_{\vec{\nu}}|},$$

where $\overline{|\alpha|} = \max_{v|_{\infty}} |\alpha|_v$ (in other words, $\overline{|\alpha|}$ is the maximum of absolute values of α 's conjugates).

It follows from (2.1)–(2.2) that $\mathcal{L}(v_n) \leq \exp(O(n \log n))$. Hence for archimedean $v \in \mathcal{M}_{\mathbb{K}}$ we have

$$|v_n(q, \cdot)|_v \leq \max\{|q|_v, 1\}^{dn^2/2 + O(n)} \exp(O(n \log n)).$$

This implies that

$$|L_{l,n}|_v \leq \max\{|q|_v, 1\}^{dn^2/2+O(n^{3/2})} \exp(O(n \log n)).$$

Taking into account the estimate

$$\prod_{v \nmid \infty} |I_n|_v^{-\kappa_v/\kappa} = \prod_{v \mid \infty} |I_n|_v^{\kappa_v/\kappa} \leq \overline{|I_n|} \leq \exp(O(n \log n))$$

and recalling that $|q|_v \leq 1$ for all $v \neq w$, we obtain condition 2 of the proposition.

Condition 3 follows from Lemma 3. Indeed, if $v_{l_0,n}(\vec{\omega}) = 0$ for all n with $n_0 \leq n \leq N_0$, then we have $V_n(q, \vec{\omega}) = 0$ for $n_0 + 1 \leq n \leq N_0/2 + 1$.

This concludes the proof of the proposition in the first case.

3.2 Case (b)

Without loss of generality, we can assume that $Q(x) = 1$.

We keep the notation from Subsection 2.2. Let us show that we can take

$$L_{l,n}(\vec{x}) = q^{-\text{ord}_z v_{l,n}} v_{l,n}(q, \vec{x}).$$

Suppose $\vec{\omega} = (\omega_0, \vec{\omega}_1) = (\omega_0, \omega_{j,k,\sigma}) \in \mathbb{C}_w^{1+\sum_{j,k} s_{j,k}}$ satisfies

$$\omega_0 + \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) = 0.$$

This implies that

$$|v_n(q, \vec{\omega})|_w = \left| - \sum_{j=n+1}^{\infty} \frac{u_j(q, \vec{\omega})}{\prod_{k=n+1}^j P(k, q^k)/Q(k)} \right|_w \leq |\vec{\omega}_1|_w |q|_w^{O(n)}.$$

Therefore,

$$|v_{l,n}(q, \vec{\omega})|_w \leq |\vec{\omega}_1|_w |q|_w^{(m+\varepsilon_0)g_2 l^3/3+O(n)},$$

and condition 1 of the proposition follows from Lemma 2.

Further, we have

$$\begin{aligned} I_n v_{l,n} &\in \mathbb{Z}_{\mathbb{K}}[z, \vec{x}], \\ \deg_z v_{l,n} &\leq dn^2/2 + O(n^{3/2}). \end{aligned}$$

Therefore, if $v \nmid \infty$, then

$$|L_{l,n}|_v \leq |I_n|_v^{-1} \max\{|q|_v, 1\}^{dn^2/2+O(n^{3/2})}.$$

Moreover,

$$\mathcal{L}(v_{l,n}) \leq \exp(O(n \log n)),$$

hence for $v|\infty$ we have

$$|L_{l,n}|_v \leq \max\{|q|_v, 1\}^{dn^2/2+O(n^{3/2})} \exp(O(n \log n)).$$

These estimates imply condition 2 of the proposition.

Condition 3 follows from Lemma 3.

Proposition 1 is proved.

4 Proof of Theorem 1

Take $n_0 = mg_0 l^2 + Sl$, where $l \in \mathbb{Z}_{>0}$ will be chosen later. It follows from Proposition 1 that there is an integer n with $mg_0 l^2 + O(l) \leq n \leq mg_0^2 l^2 + O(l)$ such that $L_{l,n}(\vec{A}) \neq 0$. By the product formula, we have

$$|L_{l,n}(\vec{A})|_w^{\varkappa_w/\varkappa} = \prod_{v \in \mathcal{M}_{\mathbb{K}} \setminus \{w\}} |L_{l,n}(\vec{A})|_v^{-\varkappa_v/\varkappa} \geq \prod_{v \in \mathcal{M}_{\mathbb{K}} \setminus \{w\}} (|L_{l,n}|_v |\vec{A}|_v)^{-\varkappa_v/\varkappa} = |\vec{A}|_w^{\varkappa_w/\varkappa} (H_w(L_{l,n}) H(\vec{A}))^{-1},$$

therefore,

$$|L_{l,n}(\vec{A})|_w \geq |\vec{A}|_w (H_w(L_{l,n}) H)^{-\varkappa/\varkappa_w} \geq |\vec{A}|_w H^{-\varkappa/\varkappa_w} |q|_w^{O(l^2 \log l)}.$$

Let $\vec{\omega} = (\omega_0, \omega_{j,k,\sigma})$ be given by

$$\begin{aligned} \omega_{j,k,\sigma} &= A_{j,k,\sigma}, \\ \omega_0 &= - \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} \omega_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k). \end{aligned}$$

It follows from Proposition 1 that

$$|L_{l,n}(\vec{\omega})|_w \leq |\vec{A}|_w |q|_w^{-ln+mg_0 l^3/2+O(n)} \leq |\vec{A}|_w |q|_w^{-mg_0 l^3/2+O(l^2)}.$$

Take $l = \left\lfloor \left(\frac{3 \log H}{mg_0 \log H(q)} \right)^{1/3} \right\rfloor$. Then we have

$$|L_{l,n}(\vec{\omega})|_w \leq \frac{1}{2} |L_{l,n}(\vec{A})|_w,$$

provided H_0 (and hence H) is sufficiently large. Therefore,

$$|L_{l,n}(\vec{A}) - L_{l,n}(\vec{\omega})|_w \geq \frac{1}{2} |L_{l,n}(\vec{A})|_w \geq \frac{1}{2} |\vec{A}|_w (H_w(L_{l,n}) H)^{-\varkappa/\varkappa_w}.$$

On the other hand, we have

$$|L_{l,n}(\vec{A}) - L_{l,n}(\vec{\omega})|_w \leq |L_{l,n}|_w |A_0 - \omega_0|_w.$$

Thus,

$$\left| A_0 + \sum_{j=1}^m \sum_{k=0}^{d_0-1} \sum_{\sigma=0}^{s_{j,k}-1} A_{j,k,\sigma} f^{(\sigma)}(\alpha_j q^k) \right|_w = |A_0 - \omega_0|_w \geq \frac{1}{2} |\vec{A}|_w (H(L_{l,n})H)^{-\varkappa/\varkappa_w},$$

and Theorem 1 follows from Proposition 1.

References

- [1] J.-P. BÉZIVIN, “Indépendance linéaire des valeurs des solutions transcendentes de certaines équations fonctionnelles II”, *Acta Arith.* **55**:3 (1990), 233–240.
- [2] I. ROCHEV, “Linear independence of values of certain q -series” (in Russian), *Izv. RAN. Ser. Mat.* **75**:1 (2011), 181–224; <http://mi.mathnet.ru/eng/izv4062>.
- [3] I. ROCHEV, “New linear independence measures for values of q -hypergeometric series”, preprint, available at [arXiv:1006.5413v1](https://arxiv.org/abs/1006.5413v1).
- [4] O. SANKILAMPI, K. VÄÄNÄNEN, “On the values of Heine series at algebraic points”, *Results Math.* **50**:1–2 (2007), 141–153; DOI: 10.1007/s00025-006-0240-2.